Causal diagrams and the identification of causal effects

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Overview

Causal diagrams

Interventions in Markovian models

Confounding bias

Intervention calculus

Identifiability tests

An example from Transport domain
Causal diagrams – Causal effect

Figure 3.1: A causal diagram representing the effect of fumigants ($X$) on yields ($Y$).

\[ Z_0 = f_0(\epsilon_0), \quad B = f_B(Z_0, \epsilon_B), \]
\[ Z_1 = f_1(Z_0, \epsilon_1), \quad X = f_X(Z_0, \epsilon_X), \]
\[ Z_2 = f_2(X, Z_1, \epsilon_2), \quad Y = f_Y(X, Z_2, Z_3, \epsilon_Y), \]
\[ Z_3 = f_3(B, Z_2, \epsilon_3). \]

\[
P(x_1, ..., x_n) = \prod_i P(x_i \mid pa_i), \quad (3.5)
\]
\[
P(z_0, x, z_1, b, z_2, z_3, y) = P(z_0)P(x\mid z_0)P(z_1\mid z_0)
\times P(b\mid z_0)P(z_2\mid x, z_1)
\times P(z_3\mid z_2, b)P(y\mid x, z_2, z_3). \quad (3.6)
\]
Causal effect

Given two disjoint sets of variables, $X$ and $Y$, the causal effect of $X$ on $Y$, denoted either as $P(y|x)$ or as $P(y|do(x))$, is a function from $X$ to the space of probability distributions on $Y$.

For each realization $x$ of $X$, $P(y|x)$ gives the probability of $Y = y$ induced by deleting from the model of (3.4) all equations corresponding to variables in $X$ and substituting $X = x$ in the remaining equations.

$$P(y \mid x^\wedge) = \sum_{Z_1} \sum_{Z_2} \sum_{Z_3} P(y \mid z_2, z_3, x) P(z_2 \mid z_1, x) \times \sum_{z_3} P(z_3 \mid z_1, z_2, x') P(z_1, x')$$
Interventions as variables

\[ P(x_i \mid pa_i) = \begin{cases} 
   P(x_i \mid pa_i) & \text{if } F_i = \text{idle,} \\
   0 & \text{if } F_i = \text{do}(x_i') \\
   1 & \text{if } F_i = \text{do}(x_i') \text{ and } x_i \neq x_i'. 
\end{cases} \]

\[ P(x_1, \ldots, x_n \mid \tilde{x}_i') = P'(x_1, \ldots, x_n \mid F_i = \text{do}(x_i')) \]

where \( P' \) is represented by \( G' \)
Computing the effect of interventions

Truncated factorization formula

\[
P(x_1, \ldots, x_n | \tilde{x}_i') = \begin{cases} 
\prod_{j \neq i} P(x_j | pa_j) & \text{if } x_i = x_i', \\
0 & \text{if } x_i \neq x_i'. 
\end{cases}
\]

\[
P(x_1, \ldots, x_n) = \prod P(x_i | pa_i),
\]

\[
P(x_1, \ldots, x_n | \tilde{x}_i') = \begin{cases} 
\frac{P(x_1, \ldots, x_n)}{P(x_i' | pa_i)} & \text{if } x_i = x_i', \\
0 & \text{if } x_i \neq x_i'. 
\end{cases}
\]

\[
P(x_1, \ldots, x_n) = \prod P(x_i | pa_i).
\]
Adjustments for direct cause

Let $PA_i$ denote the set of direct causes of variable $X_i$, and let $Y$ be any set of variables disjoint of $\{X_i \cup PA_i\}$. The effect of the intervention $do(X_i = x'_i)$ on $Y$ is given by

$$P(y|\bar{x}'_i) = \sum_{pa_i} P(y|x'_i, pa_i)P(pa_i), \quad (3.13)$$

where $P(y|x'_i, pa_i)$ and $P(pa_i)$ represent preintervention probabilities.

Compute $P(y|\bar{x}'_i)$???
Identifiability

Let $Q(M)$ be any computable quantity of a model $M$. We say that $Q$ is identifiable in a class $M$ of models if, for any pairs of models $M_1$ and $M_2$ from $M$, $Q(M_1) = Q(M_2)$ whenever $P_{M_1}(v) = P_{M_2}(v)$.

If our observations are limited, and permit only a partial set $F_M$ of features (of $P_M(v)$) to be estimated, we define $Q$ to be identifiable from $F_M$ if $Q(M_1) = Q(M_2)$ whenever $F_{M_1} = F_{M_2}$.
Causal-effect identifiability

The **causal effect** of $X$ on $Y$ is said to be **identifiable** from a graph $G$ if the quantity $P(y|\hat{x})$ can be computed uniquely from any positive probability of the observed variables—that is, if $P_{M_1}(y|\hat{x}) = P_{M_2}(y|\hat{x})$ for every pair of models $M_1$ and $M_2$ with $P_{M_1}(v) = P_{M_2}(v) > 0$ and $G(M_1) = G(M_2) = G$.

$$P(y|\hat{x}_i) = \sum_{pa_i} P(y|x_i, pa_i) P(pa_i), \quad \text{(3.13)}$$

Given a causal diagram $G$ of any Markovian model in which a subset $V$ of variables are measured, the causal effect $P(y|\hat{x})$ is identifiable whenever $\{X \cup Y \cup PA_X\} \subseteq V$, that is, whenever $X$, $Y$, and all parents of variables in $X$ are measured. The expression of $P(y|\hat{x})$ is then obtained by adjusting for $PA_x$, as in (3.13).

Given the causal diagram $G$ of any Markovian model in which all variables are measured, the causal effect $P(y|\hat{x})$ is identifiable for every two subsets of variables $X$ and $Y$ and is obtained from the truncated factorization of (3.14).

$$P(x_1, \ldots, x_n|\hat{s}) = \begin{cases} \prod_{i: x_i \notin S} P(x_i|pa_i) & \text{for } x_1, \ldots, x_n \text{ consistent with } s, \\ 0 & \text{otherwise.} \end{cases} \quad \text{(3.14)}$$
Controlling confounding bias – Back-Door Criterion

A set of variables $Z$ satisfies the back-door criterion relative to an ordered pair of variables $(X_i, X_j)$ in a DAG $G$ if:

(i) no node in $Z$ is a descendant of $X_i$; and

(ii) $Z$ blocks every path between $X_i$ and $X_j$ that contains an arrow into $X_i$.

Back-Door Adjustment

If a set of variables $Z$ satisfies the back-door criterion relative to $(X, Y)$, then the causal effect of $X$ on $Y$ is identifiable and is given by the formula

$$P(y|\hat{x}) = \sum_z P(y|x, z)P(z).$$
Controlling confounding bias – Front-Door Criterion

A set of variables $Z$ is said to satisfy the **front-door** criterion relative to an ordered pair of variables $(X, Y)$ if:

(i) $Z$ intercepts all directed paths from $X$ to $Y$;

(ii) there is no back-door path from $X$ to $Z$; and

(iii) all back-door paths from $Z$ to $Y$ are blocked by $X$.

Front-Door Adjustment

If $Z$ satisfies the front-door criterion relative to $(X, Y)$ and if $P(x, z) > 0$, then the causal effect of $X$ on $Y$ is identifiable and is given by the formula

$$P(y|x) = \sum_z P(z|x) \sum_{x'} P(y|x', z) P(x').$$
Intervention calculus – Rules of $d$ Calculus

Let $G$ be the directed acyclic graph associated with a causal model as defined in (3.2), and let $P(\cdot)$ stand for the probability distribution induced by that model. For any disjoint subsets of variables $X, Y, Z$, and $W$ we have the following rules.

**Rule 1** (Insertion/deletion of observations):

$$P(y|\tilde{x}, z, w) = P(y|\tilde{x}, w) \quad \text{if} \quad (Y \perp Z|X, W)_{G_X}.$$

**Rule 2** (Action/observation exchange):

$$P(y|\tilde{x}, \tilde{z}, w) = P(y|\tilde{x}, z, w) \quad \text{if} \quad (Y \perp Z|X, W)_{G_{XZ}}.$$

**Rule 3** (Insertion/deletion of actions):

$$P(y|\tilde{x}, \tilde{z}, w) = P(y|\tilde{x}, w) \quad \text{if} \quad (Y \perp Z|X, W)_{G_X, \overline{Z(W)}},$$

where $Z(W)$ is the set of $Z$-nodes that are not ancestors of any $W$-node in $G_X$. 
Implications of $d$-Calculus

A causal effect $q = P(y_1, ..., y_k | \hat{x}_1, ..., \hat{x}_m)$ is identifiable in a model characterized by a graph $G$ if there exists a finite sequence of transformations, each conforming to one of the inference rules in Theorem 3.4.1, that reduces $q$ into a standard (i.e. “hat”-free) probability expression involving observed quantities.
Notations of *do* Calculus

![Diagram of do calculus notations](image)

**Figure 3.6:** Subgraphs of $G$ used in the derivation of causal effects.

- $G_{\overline{X}}$: remove arrows pointing to $X$
- $G_{\overline{X}}$: remove arrows emanating from $X$
- $G_{XZ}^{-}$: remove ears of $X$ and legs of $Z$

$$P(y | \overline{x}, z) \triangleq \frac{P(y, z | \overline{x})}{P(z | \overline{x})}$$
Graphical tests of Identifiability

Identifying models
Non-identifying models

Figure 3.7: (a) A bow pattern: a confounding arc embracing a causal link $X \rightarrow Y$, thus preventing the identification of $P(y|x)$ even in the presence of an instrumental variable $Z$, as in (b). (c) A bowless graph that still prohibits the identification of $P(y|x)$. 


Identifying models

Figure 3.8: Typical models in which the effect of $X$ on $Y$ is identifiable. Dashed arcs represent confounding paths, and $Z$ represents observed covariates.
Non-identifying models

![Diagram of non-identifying models](image)

**Figure 3.9:** Typical models in which $P(y|\tilde{x})$ is not identifiable.
Excursions $\sim [f(\text{Socio-demographics}) + f(\text{Mobility tool ownership}) + f(\text{Land-use type}) + f(\text{Survey characteristics})]$
Simple regression model

\[
\text{for } (i \text{ IN } 1 : K) \\
\text{socio-demo}[i] \rightarrow \text{age}[i] \rightarrow \text{gender}[i] \rightarrow \text{hh-income}[i] \rightarrow \text{working}[i] \rightarrow \text{hh-size}[i] \rightarrow \text{car}[i] \\
\text{survey-challenge}[i] \rightarrow \text{per-inf}[i] \rightarrow \text{rep-prd}[i] \rightarrow \text{trips}[i] \rightarrow \text{tfc}[i] \rightarrow \text{excursions}[i] \\
\text{f.working} \rightarrow \text{f.hh-income} \rightarrow \text{f.gender} \rightarrow \text{f.age} \rightarrow \text{f.hh-size} \rightarrow \text{f.per-inf} \rightarrow \text{f.rep-prd} \rightarrow \text{f.car} \rightarrow \text{f.PT-tickets} \rightarrow \text{f.tfc}
\]
Random-effect regression model
Random effect regression model for multi-source data
Adequacy of the model

Model complexity

Model with lease number of free (in-effective) parameters is the least complex

Model fit

Model with maximum likelyhood function (Deviance information criterion – DIC) fits the best